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Dirichlet problem with discontinuous nonlinearities super-linear or asymptotically linear at infinity ☆

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Abstract

In the present paper, a class of Dirichlet problem with discontinuous nonlinearities super-linear or asymptotically linear at infinity is studied, and some new existence theorems of solutions are obtained.

Keywords: Critical point; Locally Lipschitz function; Generalized gradient; (C) condition

1. Introduction

Throughout the paper, we assume $f : \bar{\Omega} \times R \to R$ be a locally bounded measurable function. Consider the following Dirichlet problem:

$$\begin{cases} -\Delta u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset R^N$ is a bounded smooth domain.

As $f(x,t) \in C(\bar{\Omega} \times R)$ is asymptotically linear, not super-linear, with respect to t at infinity, Zhou [8] researched the existence of positive solutions of problem (1.1) and obtained the following theorem.

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Theorem 1.1. *If*

(H_1) $f(x,t) \in C(\bar{\Omega} \times R)$, $f(x,0) = 0$, $\forall x \in \bar{\Omega}$, $f(x,t) \geqslant 0$, $\forall t \geqslant 0$, $x \in \bar{\Omega}$ *and* $f(x,t) \equiv 0$, $\forall t \leqslant 0$, $x \in \bar{\Omega}$;
(H_2) *for a.e.* $x \in \Omega$, $\frac{f(x,t)}{t}$ *is nondecreasing with respect to* $t \geqslant 0$;
(H_3) $\lim_{t\to 0} \frac{f(x,t)}{t} = p(x)$, $\lim_{t\to+\infty} \frac{f(x,t)}{t} = q(x) \not\equiv 0$ *uniformly in a.e.* $x \in \Omega$, *where* $0 \leqslant p(x), q(x) \in L^\infty(\Omega)$ *and* $\|p\|_\infty < \lambda_1$ ($\lambda_1 > 0$ *is the first eigenvalue of* $(-\Delta, H_0^1(\Omega))$), *then*

(1) *there is no positive solution to the problem* (1.1) *if* $\Lambda > 1$;
(2) *the problem* (1.1) *has a positive solution if* $\Lambda < 1$;
(3) *if* $\Lambda = 1$, *then problem* (1.1) *has a positive solution* $u \in H_0^1(\Omega)$ *if and only if there exists a constant* $c > 0$ *such that* $u(x) = c\varphi_\Lambda(x)$ *and* $f(x,u) = q(x)u$ *a.e. in* $x \in \Omega$, *where* $\varphi_\Lambda(x) > 0$ *is the function which achieves* $\Lambda = \inf\{\int_\Omega |\nabla u|^2\,dx\colon\ u \in H_0^1(\Omega), \int_\Omega q(x)u^2\,dx = 1\}$ (*see also* [4]).

As $f(x,t) \in C(\bar{\Omega} \times R)$ is super-linear with respect to t at infinity, Zhou [8] obtained the following theorem.

Theorem 1.2. *If* (H_1), (H_2) *and* (H_3) *with* $q(x) = +\infty$ *and* $\lim_{t\to+\infty} \frac{f(x,t)}{t^{r-1}} = 0$ *uniformly in* $x \in \Omega$ *for some* $r \in (2, \frac{2N}{N-2})$ *if* $N > 2$, *or* $r \in (2, +\infty)$ *if* $N = 1, 2$, *then problem* (1.1) *has at least one positive solution* $u \in H_0^1(\Omega)$.

In 1976, Clarke [2] introduced the generalized gradients. In 1981, Chang [3] studied variational methods for nondifferentiable functionals by the generalized gradients and obtained several critical point theorems. As applications of these critical point theorems, Chang [3] researched partial differential equations with discontinuous nonlinearities. Since then, the results of Chang [3] become basic tools for the study of differential equation with discontinuous nonlinearities. Recently, Kyritsi and Papageorgiou [5] and Wu [7] studies critical points theory of nonsmooth function under nonsmooth C-condition, Bonanno and Giovannelli [1] researched nonsmooth eigenvalue problems and Marano and Motreanu [6] researched nonsmooth nonlinear boundary value problems under nonsmooth P.S.-condition. And as applications, Kyritsi and Papageorgiou [5] and Wu [7] researched the existence of solutions for hemivariational inequalities.

In the present paper, our main purpose is to study the existence of positive solution for problem (1.1) with $f : \bar{\Omega} \times R \to R$ be a locally bounded measurable function. Our main results contain Theorems 1.1 and 1.2 as their special cases and are different from these results in [1,5–7].

2. Preliminaries

Let X be a real Banach space and X^* the dual space of X. We denote by $\langle\cdot,\cdot\rangle$ the duality pairing between X^* and X. A functional $J : X \to R$ is called locally Lipschitz if for each $u \in X$ there exist a neighborhood U of u and a constant $L \geqslant 0$ such that

$$\bigl|J(v) - J(w)\bigr| \leqslant L\|v - w\|, \quad \forall v, w \in U.$$

For any $u, v \in X$, we define the generalized directional derivative $J^0(u; v)$ of J at point u along the direction v as

$$J^0(u; v) = \overline{\lim_{h\to 0,\ \lambda\downarrow 0}} \frac{1}{\lambda}\big[J(u+h+\lambda v) - J(u+h)\big].$$

The generalized gradient of the function J at u, denoted by $\partial J(u)$, is the set

$$\partial J(u) = \big\{w \in X^*\colon \langle w, v\rangle \leqslant J^0(u; v),\ \forall v \in X\big\}.$$

Set

$$\lambda(u) = \min_{w\in\partial J(u)} \|w\|.$$

A point $x_0 \in X$ is said to be a critical point of J if zero element in X^* $\theta \in \partial J(x_0)$. See [2,3] for the properties of the generalized directional derivative and the generalized gradient.

We need the following concept, which can be find in [5].

Definition 2.1. Let X be a normed linear space and $f : X \to R$ a locally Lipschitz function. We say that f satisfies the $(C)_c$ condition, if any sequence $\{x_n\} \subset X$ along which $f(x_n) \to c$ and $(1 + \|x_n\|)\lambda(x_n) \to 0$ possesses a convergent subsequence.

The following theorem is our main tool, which is Theorem 2.5 in [7].

Theorem 2.1. *Let X be a reflexive Banach space and $f : X \to R$ a locally Lipschitz function. Assume that there exist a neighborhood U of 0, a point $x_0 \notin U$ and a constant β such that*

$$f(0), \qquad f(x_0) < \beta, \qquad f|_{\partial U} \geqslant \beta.$$

Let $\Gamma = \{\varphi \in C([0,1], X)\colon \varphi(0) = 0,\ \varphi(1) = x_0\}$ and $c = \inf_{\varphi\in\Gamma} \max_{t\in[0,1]} f(\varphi(t))$. Then $c \geqslant \beta$ and there exists a sequence $\{x_n\} \subset X$ such that $f(x_n) \to c$ and

$$\big(1 + \|x_n\|\big)\lambda(x_n) \to 0.$$

Furthermore, if f satisfies the $(C)_c$ condition, then c is a critical value of f.

3. Main results

Let $\Omega \subset R^N$ be a nonempty bounded open subset with a smooth boundary $\partial\Omega$. Given one locally bounded measurable function $f : \bar{\Omega} \times R \to R$. f is called superposition measurable if $u : \Omega \to R$ is measurable implies that $x \mapsto f(x, u(x))$ is measurable. Set

$$f^-(x,t) = \lim_{\delta\to 0} \inf_{|\xi - t|<\delta} f(x,\xi), \qquad f^+(x,t) = \lim_{\delta\to 0} \sup_{|\xi - t|<\delta} f(x,\xi).$$

Then $f^-(x,\cdot)$ and $f^+(x,\cdot)$ are, respectively, lower semi-continuous and upper semi-continuous.

Consider the following problems:

Find $u \in H_0^1(\Omega)$ such that

$$\begin{cases} -\Delta u \in [f^-(x,u(x)), f^+(x,u(x))], & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{P$_1$}$$

or

$$\begin{cases} -\Delta u = f(x,u(x)), & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{P$_2$}$$

where f satisfies the following conditions:

(A_1) $f(x,0)\equiv 0$ for all $x\in\bar{\Omega}$ and

$$f(x,t)\begin{cases}\geqslant(\not\equiv)0, & \text{if } t\geqslant 0,\\ \equiv 0, & \text{if } t\leqslant 0;\end{cases}$$

(A_2) $\frac{f(x,t)}{t}$ is nondecreasing in $t\geqslant 0$ for a.e. $x\in\Omega$;

(A_3) $\lim_{t\to 0}\frac{f(x,t)}{t}=m(x)$, $\lim_{t\to\infty}\frac{f(x,t)}{t}=q(x)\not\equiv 0$, where $\|m\|_\infty<\lambda_1$, λ_1 is the first eigenvalue of the eigenvalue problem

$$\begin{cases}-\Delta u=\lambda u, & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega;\end{cases}$$

(A_4) the functions f^- and f^+ are superposition measurable;

(A_5) there exists a constant r with

$$r\in\begin{cases}(2,2^*), & \text{if } N>2,\\ (2,+\infty), & \text{if } N=1,2\end{cases}$$

such that

$$\lim_{t\to+\infty}\frac{f(x,t)}{t^{r-1}}=0$$

uniformly for all $x\in\Omega$;

(A_6) there exists a set $\Omega_f\subset\Omega$ with $|\Omega_f|$ (the measure of Ω_f) $=0$ such that the set

$$D_f:=\bigcup_{x\in(\Omega\setminus\Omega_f)}\bigl\{t\in R\colon\ f(x,\cdot)\text{ is discontinuous at } t\bigr\}$$

has measure zero, and for almost every $x\in\Omega$ each $t\in D_f$ the condition $f^-(x,t)\leqslant 0\leqslant f^+(x,t)$ implies $f(x,t)=0$.

Let Λ is the first eigenvalue of the eigenvalue problem

$$\begin{cases}-\Delta u=\lambda q(x)u, & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega.\end{cases}$$

Then $\Lambda>0$ is simple, isolated, $\Lambda=\inf\{\int_\Omega|\nabla u|^2\,dx\colon\ u\in H_0^1(\Omega)$ and $\int_\Omega q(x)|u|^2\,dx=1\}$ and the infimum Λ can be achieved at some $\varphi_\Lambda>0$ with $\int_\Omega q(x)|\varphi_\Lambda|^2\,dx=1$ (see [4]).

Theorem 3.1. (i) *If conditions* (A_1) *to* (A_4) *hold,* $q(x)\in L^\infty(\Omega)$ *and* $\Lambda>1$*, then the problem* (P_1) *has no any positive solution in* $H_0^1(\Omega)$.

(ii) *If conditions* (A_1), (A_3), (A_4) *hold,* $q(x)\in L^\infty(\Omega)$ *and* $\Lambda<1$*, then the problem* (P_1) *possesses at least one positive solution in* $H_0^1(\Omega)$*. If adds the condition* (A_6)*, then the problem* (P_2) *possesses at least one positive solution in* $H_0^1(\Omega)$.

(iii) *If conditions* (A_1) *to* (A_4) *hold,* $q(x)\in L^\infty(\Omega)$ *and* $\Lambda=1$*, then the problem* (P_1) *possesses one positive solution* $u\in H_0^1(\Omega)\Leftrightarrow$ *there exists a positive constant* α *such that* $u=\alpha\varphi_\Lambda$ *and* $f^+(x,u(x))=q(x)u(x)$ *for a.e.* $x\in\Omega$.

Proof. Let $X=H_0^1(\Omega)$, $Y=L^2(\Omega)$, and

$$J(u)=\frac{1}{2}\int_\Omega|\nabla u|^2\,dx-\int_\Omega F\bigl(x,u(x)\bigr)\,dx,\quad \forall u\in L^2(\Omega),$$

where $F(x,t)=\int_0^t f(x,s)\,ds$. Then the Sobolev embedding $H_0^1(\Omega)\hookrightarrow L^2(\Omega)$ is compact and J is well defined and locally Lipschitz in X.

Step 1. There exist constants $\rho,\beta>0$ such that $J(u)\geqslant\beta$ whenever $u\in H_0^1(\Omega)$ and $\|u\|=\rho$.

Indeed, taking $2<r<2^*$. For any $0<\varepsilon<\lambda_1-\|m\|_\infty$, by ($A_1$) and ($A_3$), there exists a negative function $\alpha(x)\in L^\infty(\Omega)$ such that

$$\bigl|f(x,t)\bigr|<\bigl(m(x)+\varepsilon\bigr)|t|+\alpha(x)|t|^{r-1}$$

for all $x\in\Omega$ and all $t\in R$. Consequently,

$$\begin{aligned}
J(u)&=\frac{1}{2}\|u\|^2-\int_\Omega F(x,u)\,dx\\
&\geqslant\frac{1}{2}\left(1-\frac{\|m\|_\infty+\varepsilon}{\lambda_1}\right)\|u\|^2-\frac{1}{r}\int_\Omega\alpha(x)|u|^r\,dx\\
&\geqslant\frac{1}{2}\left(1-\frac{\|m\|_\infty+\varepsilon}{\lambda_1}\right)\|u\|^2-\frac{1}{r}\|\alpha\|_s\|u\|^r_{\frac{rs}{s-1}}\\
&\geqslant\frac{1}{2}\left(1-\frac{\|m\|_\infty+\varepsilon}{\lambda_1}\right)\|u\|^2-c_1\|u\|^r.
\end{aligned}$$

Since $r>2$ and $\frac{1}{2}(1-\frac{\|m\|_\infty+\varepsilon}{\lambda_1})>0$, there exists small $\rho>0$ such that

$$J(u)\geqslant\frac{1}{4}\left(1-\frac{\|m\|_\infty+\varepsilon}{\lambda_1}\right)\rho^2:=\beta$$

whenever $u\in H_0^1(\Omega)$ and $\|u\|=\rho$.

Step 2. If $\Lambda<1$, then $\lim_{t\to+\infty}J(t\varphi_\Lambda)=-\infty$. If $q(x)=+\infty$, then $\lim_{t\to+\infty}J(t\varphi_1)=-\infty$, where $\varphi_1>0$ is the eigenfunction corresponding to λ_1.

The proofs are similar to the proofs of Lemma 2.2(b) in [8].

Step 3. Proof of (i).

If $u\in H_0^1(\Omega)$ is a positive solution of the problem (P_1), then

$$\begin{aligned}
\int_\Omega|\nabla u|^2\,dx&\leqslant\int_\Omega f^+\bigl(x,u(x)\bigr)u(x)\,dx=\int_\Omega\lim_{\delta\to0}\sup_{|\xi-u(x)|<\delta}f(x,\xi)u(x)\,dx\\
&\leqslant\int_\Omega\lim_{\delta\to0}\sup_{|\xi-u(x)|<\delta}q(x)|\xi|u(x)\,dx=\int_\Omega q(x)|u|^2\,dx.
\end{aligned}$$

This shows $\Lambda\leqslant1$. Hence, if $\Lambda>1$, then the problem (P_1) has no any positive solution in $H_0^1(\Omega)$.

Step 4. Proof of (ii).

Since $\Lambda < 1$, by the conclusion of Step 2, there exists large $t_0 > 0$ such that $\|t_0\varphi_\Lambda\| > \rho$ and $J(t_0\varphi_\Lambda) < 0$. Set $M = [0,1]$, $M^* = \{0,1\}$, $\gamma^*(t) = tt_0\varphi_\Lambda$, $\forall t \in M$, and

$$\Gamma = \big\{\gamma \in c(M,X)\colon \gamma(0) = 0,\ \gamma(1) = t_0\varphi_\Lambda\big\}, \quad c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J\big(\gamma(t)\big).$$

Then $c \geqslant \beta > 0$. Now, we prove J satisfies the $(C)_c$ condition. Indeed, if any sequence $\{u_n\} \subset H_0^1(\Omega)$ such that $J(u_n) \to c$ and $(1 + \|u_n\|)\lambda(u_n) \to 0$, we shall prove that $\{u_n\}$ has a convergent subsequence.

Since $\lambda(u_n) = \min_{x^*\in\partial J(u_n)} \|x^*\|_{X^*}$, there exists $x_n^* \in \partial J(u_n)$ such that $\lambda(u_n) = \|x_n^*\|$. Hence

$$\big|\langle x_n^*, u_n\rangle\big| \leqslant \lambda(u_n)\big(1 + \|u_n\|\big) \to 0.$$

Since $x_n^* \in \partial J(u_n)$, there exists $z_n(x) \in [f^-(x,u_n(x)), f^+(x,u_n(x))]$ such that

$$\langle x_n^*, \varphi\rangle = \int_\Omega \nabla u_n \cdot \nabla\varphi\,dx - \int_\Omega z_n(x)\varphi(x)\,dx, \quad \forall\varphi \in X,$$

and hence

$$\|u_n\|^2 - \int_\Omega z_n(x)u_n(x)\,dx \to 0. \tag{3.1}$$

We claim that the sequence $\{u_n\}$ is bounded in X. Otherwise, we can assume $\|u_n\| \to \infty$. Set $w_n = \frac{u_n}{\|u_n\|}$. Then $\{w_n\}$ is bounded in X. Since X is a reflexive Banach space, we can assume that there exists $w \in X$ such that

$$w_n \rightharpoonup w \quad \text{in } X, \qquad w_n \to w \quad \text{in } Y, \quad \text{and} \quad w_n(x) \to w(x) \quad \text{for a.e. } x \in \Omega.$$

If $w = 0$, then, by (A_1), (A_3) and (3.1),

$$\begin{aligned} 1 &= \lim_{n\to\infty} \frac{1}{\|u_n\|^2}\int_\Omega z_n(x)u_n(x)\,dx = \lim_{n\to\infty}\int_\Omega \frac{z_n}{u_n}w_n^2\,dx \\ &\leqslant \lim_{n\to\infty}\int_\Omega \frac{f^+(x,u_n(x))}{u_n}w_n^2\,dx \leqslant c_1 \lim_{n\to\infty}\int_\Omega w_n^2\,dx = 0. \end{aligned}$$

This is a contradiction. Hence $w \neq 0$. Consequently, for a.e. $x \in \Omega$, $u_n(x) \to +\infty$ as $n \to \infty$, and hence $\lim_{n\to\infty} \frac{f(x,u_n(x))}{u_n(x)} = q(x)$ for a.e. $x \in \Omega$. Consequently, by (A_3), we can prove

$$\lim_{n\to\infty} \frac{f^+(x,u_n(x))}{u_n(x)} = q(x), \qquad \lim_{n\to\infty} \frac{f^-(x,u_n(x))}{u_n(x)} = q(x),$$

and so, $\lim_{n\to\infty} \frac{z_n(x)}{u_n(x)} = q(x)$. Set

$$p_n(x) = \begin{cases} 0, & \text{if } u_n(x) \leqslant 0, \\ \frac{z_n(x)}{u_n(x)}, & \text{if } u_n(x) > 0. \end{cases}$$

By (A_1) and (A_3) there exists $M > 0$ such that $0 \leqslant p_n(x) \leqslant M$. Then for each $\varphi \in X$, one has

$$\begin{aligned} \int_\Omega p_n(x)w_n(x)\varphi(x)\,dx &= \int_\Omega p_n(x)\big[w_n(x) - w(x)\big]\varphi(x)\,dx + \int_\Omega p_n(x)w(x)\varphi(x)\,dx \\ &\to \int_\Omega q(x)w(x)\varphi(x)\,dx. \end{aligned}$$

Moreover,

$$\left|\int_\Omega \nabla w_n \cdot \nabla\varphi\, dx - \int_\Omega p_n(x)w_n(x)\varphi(x)\, dx\right| = \frac{1}{\|u_n\|}\left|\langle x_n^*, \varphi\rangle\right| \leqslant \frac{1}{\|u_n\|}\lambda(u_n)\|\varphi\| \to 0.$$

Hence

$$\int_\Omega \nabla w \cdot \nabla\varphi\, dx - \int_\Omega q(x)w(x)\varphi(x)\, dx = 0.$$

This contradicts that $\Lambda < 1$. Therefore, $\{u_n\}$ is bounded in X. Consequently, as the proof 2^0 of Theorem 4.3 in [3] we may prove that $\{u_n\}$ possess a convergent subsequence, and hence J satisfies the $(C)_c$ condition. By Theorem 2.1 we know that c is a critical value of J. Hence there exist $u \in H_0^1(\Omega)$ and $z \in [f^-(x,u), f^+(x,u)]$ such that $J(u) = c \geqslant \beta > 0$ and

$$\int_\Omega \nabla u \cdot \nabla\varphi\, dx = \int_\Omega z(x)\varphi(x)\, dx \geqslant 0$$

for all nonnegative functions $\varphi \in C_0^1(\Omega)$. Consequently, by weak maximum principle we know that $u > 0$, i.e. $u \in H_0^1(\Omega)$ is a positive solution of the problem (P_1). If adds the condition (A_6), then as the proof of corresponding parts of Theorem 3.1 in [1] or Theorem 4.2 in [6] we can prove that the problem (P_2) possesses at least one positive solution in $H_0^1(\Omega)$.

Step 5. Proof of (iii).

Since $\Lambda = 1$,

$$\int_\Omega \nabla\varphi_\Lambda \cdot \nabla\varphi\, dx - \int_\Omega q(x)\varphi_\Lambda(x)\varphi(x)\, dx = 0, \quad \forall\varphi \in H_0^1(\Omega).$$

If $u \in H_0^1(\Omega)$ is a positive solution of the problem (P_1), then there exists $z \in [f^-(x,u), f^+(x,u)]$ such that

$$\int_\Omega \nabla u \cdot \nabla\varphi\, dx = \int_\Omega z(x)\varphi(x)\, dx, \quad \forall\varphi \in H_0^1(\Omega). \tag{3.2}$$

Hence

$$\int_\Omega \nabla u \cdot \nabla\varphi_\Lambda\, dx = \int_\Omega z(x)\varphi_\Lambda(x)\, dx,$$

and hence

$$\begin{aligned}\int_\Omega q(x)\varphi_\Lambda(x)u(x)\, dx &= \int_\Omega \nabla\varphi_\Lambda \cdot \nabla u\, dx = \int_\Omega z(x)\varphi_\Lambda(x)\, dx \\ &\leqslant \int_\Omega f^+(x,u)\varphi_\Lambda(x)\, dx \leqslant \int_\Omega q(x)u(x)\varphi_\Lambda(x)\, dx.\end{aligned}$$

Therefore,

$$\int_\Omega \left[q(x)u(x) - z(x)\right]\varphi_\Lambda(x)\, dx = \int_\Omega \left[f^+(x,u) - z(x)\right]\varphi_\Lambda(x)\, dx = 0.$$

By (A_2) we know $f^+(x,u) = z(x) = q(x)u(x)$ for a.e. $x \in \Omega$. Consequently, by (3.2) one has

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx = \int_\Omega q(x)u(x)\varphi(x)\, dx, \quad \forall \varphi \in H_0^1(\Omega).$$

This shows that $u > 0$ is an eigenfunction associated to Λ. Since Λ is simple, there exists a constant $\alpha > 0$ such that $u = \alpha\varphi_\Lambda$.

Conversely, if there exists a positive constant α such that $u = \alpha\varphi_\Lambda$ and $f^+(x,u(x)) = q(x)u(x)$ for a.e. $x \in \Omega$, then

$$\begin{aligned}\int_\Omega \nabla u \cdot \nabla \varphi \, dx &= \alpha \int_\Omega \nabla \varphi_\Lambda \cdot \nabla \varphi \, dx = \alpha \int_\Omega q(x)\varphi_\Lambda(x)\varphi(x)\, dx \\ &= \int_\Omega q(x)u(x)\varphi(x)\, dx = \int_\Omega f^+\big(x,u(x)\big)\varphi(x)\, dx\end{aligned}$$

for all $\varphi \in H_0^1(\Omega)$. Hence $u = \alpha\varphi_\Lambda$ is a positive solution of the problem (P_1). □

Corollary 3.2. *Let conditions* (A_1) *to* (A_4) *hold and* $q(x) \equiv l > 0$.

(i) *If* $\lambda_1 > l$, *then the problem* (P_1) *has no any positive solution in* $H_0^1(\Omega)$;
(ii) *If* $\lambda_1 < l < +\infty$, *then the problem* (P_1) *possesses at least one positive solution in* $H_0^1(\Omega)$;
(iii) *If* $\lambda_1 = l$, *then the problem* (P_1) *possesses one positive solution* $u \in H_0^1(\Omega) \Leftrightarrow$ *there exists a positive constant* α *such that* $u = \alpha\varphi_1$ *and* $f^+(x,u(x)) = \lambda_1 u(x)$ *for a.e.* $x \in \Omega$.

Proof. Notice that $\Lambda = \frac{\lambda_1}{l}$. The conclusion follows from Theorem 3.1. □

Theorem 3.3. *Let conditions* (A_1)–(A_5) *hold and* $q(x) \equiv +\infty$. *If adds the condition* (A_6) *with*

$$\Omega_f := \bigcup_{t>0} \big\{x \in \Omega\colon\ f^+(x,t) > f^-(x,t)\big\},$$

then the problem (P_2) *possesses at least one positive solution in* $H_0^1(\Omega)$.

Proof. For any $0 < \varepsilon < \lambda_1 - \|m\|_\infty$, by (A_1), (A_3) and that

$$\lim_{t\to+\infty} \frac{f(x,t)}{t^{r-1}} = 0$$

uniformly for all $x \in \Omega$, there exist a positive constant α such that

$$\big|f(x,t)\big| < \big(m(x)+\varepsilon\big)|t| + \alpha|t|^{r-1} \tag{3.3}$$

for all $x \in \Omega$ and all $t \in R$. Hence there exist constants $\rho, \beta > 0$ such that $J(u) \geqslant \beta$ whenever $u \in H_0^1(\Omega)$ and $\|u\| = \rho$ (see Step 1 of the proof of Theorem 3.1).

Since $q(x) \equiv +\infty$, by the conclusion of Step 2 of the proof of Theorem 3.1, there exists large $t_0 > 0$ such that $\|t_0\varphi_1\| > \rho$ and $J(t_0\varphi_1) < 0$. Set $M = [0,1]$, $M^* = \{0,1\}$, $\gamma^*(t) = tt_0\varphi_1$, $\forall t \in M$, and

$$\Gamma = \big\{\gamma \in c(M,X)\colon\ \gamma(0) = 0,\ \gamma(1) = t_0\varphi_1\big\}, \quad c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J\big(\gamma(t)\big).$$

Then $c \geqslant \beta > 0$. Now, we prove J satisfies the $(C)_c$ condition. Indeed, if any sequence $\{u_n\} \subset H_0^1(\Omega)$ such that $J(u_n) \to c$ and $(1 + \|u_n\|)\lambda(u_n) \to 0$, we shall prove that $\{u_n\}$ has a convergent subsequence.

We claim that the sequence $\{u_n\}$ is bounded in X. Otherwise, we can assume $\|u_n\| \to \infty$. Set $w_n = \frac{u_n}{\|u_n\|}$. Then $\{w_n\}$, $\{w_n^+\}$ and $\{w_n^-\}$ are bounded in X. Since X is a reflexive Banach space, we can assume that there exists $w \in X$ such that

$$w_n \rightharpoonup w \quad \text{in } X, \qquad w_n \to w \quad \text{in } Y, \quad \text{and} \quad w_n(x) \to w(x) \quad \text{for a.e. } x \in \Omega,$$

and

$$w_n^+ \rightharpoonup w^+ \quad \text{in } X, \qquad w_n^+ \to w^+ \quad \text{in } Y, \quad \text{and} \quad w_n^+(x) \to w^+(x) \quad \text{for a.e. } x \in \Omega.$$

Set $\Omega_1 = \{x \in \Omega\colon w^+(x) > 0\}$. Since $q(x) = +\infty$, by (A$_1$), (A$_3$) and (3.1),

$$\begin{aligned} 1 &= \lim_{n\to\infty} \frac{1}{\|u_n\|^2} \int_\Omega z_n(x)u_n(x)\,dx = \lim_{n\to\infty} \int_\Omega \frac{z_n}{u_n} w_n^2\,dx \\ &\geqslant \lim_{n\to\infty} \int_\Omega \frac{f^-(x, u_n^+(x))}{u_n^+} (w_n^+)^2\,dx \geqslant \lim_{n\to\infty} \int_{\Omega_1} \frac{f^-(x, u_n^+(x))}{u_n^+} (w_n^+)^2\,dx = +\infty, \end{aligned}$$

whenever the measure of Ω_1, $|\Omega_1| > 0$. This is a contradiction. Hence $|\Omega_1| = 0$. Consequently, $w^+(x) = 0$ for a.e. $x \in \Omega$, and so that, by (3.3) and $r \in (2, 2^*)$,

$$\lim_{n\to+\infty} \int_\Omega F\big(x, 2\sqrt{c}\,w_n^+(x)\big)\,dx = 0.$$

Consequently,

$$\lim_{n\to+\infty} J\big(2\sqrt{c}w_n\big) = 2c.$$

Let $g(u) = \int_\Omega F(x, u(x))\,dx$ and $x_n^* \in \partial J(u_n)$ such that $\lambda(u_n) = \|x_n^*\|$. Then there exists $z_n \in \partial g(u_n)$ such that

$$\langle x_n^*, \varphi\rangle = \int_\Omega \nabla u_n \cdot \nabla\varphi\,dx - \langle z_n, \varphi\rangle, \quad \forall \varphi \in X.$$

Since

$$\big|\langle x_n^*, u_n\rangle\big| \leqslant \lambda(u_n)\big(1 + \|u_n\|\big) \to 0, \qquad \|u_n\|^2 - \langle z_n, u_n\rangle \to 0.$$

Hence we may assume that

$$-\frac{1}{n} < \|u_n\|^2 - \langle z_n, u_n\rangle < \frac{1}{n}, \quad \forall n \geqslant 1. \tag{3.4}$$

For any $t \geqslant 0$,

$$\begin{aligned} J(tu_n) &= \frac{1}{2}t^2\|u_n\|^2 - \int_\Omega F\big(x, tu_n(x)\big)\,dx \\ &< \frac{1}{2}t^2\left[\frac{1}{n} + \langle z_n, u_n\rangle\right] - \int_\Omega F\big(x, tu_n(x)\big)\,dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{t^2}{2n} + \frac{1}{2}t^2\langle z_n, u_n\rangle - \int_\Omega F\big(x, tu_n(x)\big)\,dx \\
&\leqslant \frac{t^2}{2n} + \frac{1}{2}t^2 g^0(u_n; u_n) - \int_\Omega F\big(x, tu_n(x)\big)\,dx \\
&\leqslant \frac{t^2}{2n} + \int_\Omega \left[\frac{1}{2}t^2 F^0\big(x, u_n(x); u_n(x)\big) - F\big(x, tu_n(x)\big)\right]dx \\
&= \frac{t^2}{2n} + \int_{u_n(x)>0} \left[\frac{1}{2}t^2 u_n(x) f^+\big(x, u_n(x)\big) - F\big(x, tu_n(x)\big)\right]dx.
\end{aligned}
$$

For any fixed $x \in \Omega$ and positive integer n, set

$$h(t) = \frac{1}{2}t^2 u_n^+(x) f^+\big(x, u_n^+(x)\big) - F\big(x, tu_n^+(x)\big).$$

Then the function $h(t)$ is absolutely continuous in any closed interval $[a, b] \subset [0, +\infty)$ and differentiable almost everywhere in $(0, +\infty)$ and for a.e. $t \in [0, +\infty)$, one has

$$
\begin{aligned}
\frac{d}{dt}h(t) &= tu_n^+(x) f^+\big(x, u_n^+(x)\big) - f\big(x, tu_n^+(x)\big)u_n^+(x) \\
&= t\left[u_n^+(x) f^+\big(x, u_n^+(x)\big) - \frac{f(x, tu_n^+(x))}{t}u_n^+(x)\right].
\end{aligned}
$$

Consequently, as $0 \leqslant t_1 \leqslant 1$, one has

$$
\begin{aligned}
h(1) - h(t_1) &= \int_{t_1}^{1} \frac{d}{dt}h(t)\,dt \\
&= \frac{1}{2}\big(1 - t_1^2\big)u_n^+(x) f^+\big(x, u_n^+(x)\big) - \int_{t_1}^{1} f\big(x, tu_n^+(x)\big)u_n^+(x)\,dt \\
&\geqslant \frac{1}{2}\big(1 - t_1^2\big)u_n^+(x)\big[f^+\big(x, u_n^+(x)\big) - f\big(x, u_n^+(x)\big)\big] \\
&\geqslant 0,
\end{aligned}
$$

i.e. $h(t_1) \leqslant h(1)$ for all $t_1 \in [0, 1]$. Consequently, for all $t \in [0, 1]$, one has

$$J(tu_n) \leqslant \frac{t^2}{2n} + \int_{u_n(x)>0} \left[\frac{1}{2}u_n(x) f^+\big(x, u_n(x)\big) - F\big(x, u_n(x)\big)\right]dx.$$

On the other hand,

$$
\begin{aligned}
J(u_n) &= \frac{1}{2}\|u_n\|^2 - \int_\Omega F\big(x, u_n(x)\big)\,dx \\
&> \frac{1}{2}\left[-\frac{1}{n} + \langle z_n, u_n\rangle\right] - \int_\Omega F\big(x, u_n(x)\big)\,dx
\end{aligned}
$$

$$= -\frac{1}{2n} - \frac{1}{2}\langle z_n, -u_n\rangle - \int_\Omega F\big(x, u_n(x)\big)\,dx$$
$$\geqslant -\frac{1}{2n} - \frac{1}{2}g^0(u_n; -u_n) - \int_\Omega F\big(x, u_n(x)\big)\,dx$$
$$\geqslant -\frac{1}{2n} - \frac{1}{2}\int_\Omega F^0\big(x, u_n(x); -u_n(x)\big)\,dx - \int_\Omega F\big(x, u_n(x)\big)\,dx$$
$$= -\frac{1}{2n} + \int_{u_n(x)>0} \left[\frac{1}{2}u_n(x)f^-\big(x, u_n(x)\big) - F\big(x, u_n(x)\big)\right]dx.$$

Hence, for all $t \in [0, 1]$, one has

$$J(tu_n) \leqslant \frac{1+t^2}{2n} + \int_{u_n(x)>0} \left[\frac{1}{2}u_n(x)\big[f^+\big(x, u_n(x)\big) - f^-\big(x, u_n(x)\big)\big]\right]dx + J(u_n).$$

Since $|\Omega_f| = 0$,

$$\int_{u_n(x)>0} \left[\frac{1}{2}u_n(x)\big[f^+\big(x, u_n(x)\big) - f^-\big(x, u_n(x)\big)\big]\right]dx = 0.$$

Hence

$$J(tu_n) \leqslant \frac{1+t^2}{2n} + J(u_n).$$

Consequently,

$$J\big(2\sqrt{c}w_n\big) = J\left(\frac{2\sqrt{c}}{\|u_n\|}u_n\right) \leqslant \frac{1}{2n}\left(1 + \frac{4c}{\|u_n\|^2}\right) + J(u_n)$$

and hence $2c \leqslant c$. This contradicts that $c > 0$. Hence the sequence $\{u_n\}$ is bounded in X. Consequently, as the proof 2^0 of Theorem 4.3 in [3] we may prove that $\{u_n\}$ possess a convergent subsequence, and hence J satisfies the $(C)_c$ condition. By Theorem 2.1 we know that c is a critical value of J. Hence there exist $u \in H_0^1(\Omega)$ and $z \in [f^-(x, u), f^+(x, u)]$ such that $J(u) = c \geqslant \beta > 0$ and

$$\int_\Omega \nabla u \cdot \nabla\varphi\,dx = \int_\Omega z(x)\varphi(x)\,dx \geqslant 0$$

for all nonnegative functions $\varphi \in C_0^1(\Omega)$. Consequently, by weak maximum principle we know that $u > 0$, i.e. $u \in H_0^1(\Omega)$ is a positive solution of the problem (P_1). The rest of the proof is the same with Theorem 3.1. □

Remark. Theorems 3.1 and 3.3 generalize and improve Theorems 1.1 and 1.2 to the case that $f : \bar{\Omega} \times R \to R$ is only locally bounded measurable function, respectively. Moreover, by Theorems 3.1 and 3.3 we know that the condition $f(x, t) \in C(\bar{\Omega} \times R)$ in Theorems 1.1 and 1.2 may be weakened as $f(x, t)$ is a Caratheodory function.

The following Theorem 3.4 and Corollary 3.5 are new results, too.

Theorem 3.4. *Let conditions* (A_1), (A_3)–(A_5) *hold and* $q(x) \equiv +\infty$. *If there exists* $\mu \geqslant r$ *such that*

$$\liminf_{t\to+\infty} \frac{tf(x,t)-2F(x,t)}{|t|^\mu} > 0,$$

then the problem (P_1) *possesses at least one positive solution in* $H_0^1(\Omega)$. *If adds the condition* (A_6), *then the problem* (P_2) *possesses at least one positive solution in* $H_0^1(\Omega)$.

Proof. It is sufficient to prove that J satisfies the $(C)_c$ condition from the proof of Theorem 3.3. By (A_1), (A_3) and (A_5), there exists a constant $c_1 > 0$ such that

$$0 \leqslant f(x,t) \leqslant c_1\left(1+|t|^{r-1}\right), \quad \forall t \in R.$$

Hence

$$F(x,t) = \int_0^t f(x,s)\,ds \leqslant c_1 + c_2 t^r, \quad \forall t \geqslant 0. \tag{3.5}$$

By

$$\liminf_{t\to+\infty} \frac{tf(x,t)-2F(x,t)}{|t|^\mu} > 0$$

we can prove that

$$\liminf_{t\to+\infty} \frac{tf^-(x,t)-2F(x,t)}{t^\mu} > 0. \tag{3.6}$$

Indeed, there exist $\varepsilon_0 > 0$, $M_0 > 0$ such that

$$\frac{sf(x,s)-2F(x,s)}{s^\mu} > \varepsilon_0, \quad \forall s \geqslant M_0.$$

For each fixed t with $t > M_0 + 2$, as $\delta \in (0,1)$ is enough small, one has $\frac{t}{\xi} > 0$ and

$$\xi > M_0 + 2, \qquad \frac{t\xi^\mu}{t^\mu \xi} > \frac{1}{2}, \qquad \frac{2F(x,\xi)}{t^{\mu-1}\xi} - \frac{2F(x,t)}{t^\mu} > -\frac{\varepsilon_0}{3}$$

for all ξ with $|\xi - t| < \delta$. Hence

$$\begin{aligned}\frac{tf(x,\xi)-2F(x,t)}{t^\mu} &= \frac{\frac{t}{\xi}\cdot \xi f(x,\xi) - 2F(x,t)}{t^\mu} > \frac{\frac{t}{\xi}[2F(x,\xi)+\varepsilon_0\xi^\mu] - 2F(x,t)}{t^\mu} \\ &= \frac{2F(x,\xi)}{t^{\mu-1}\xi} - \frac{2F(x,t)}{t^\mu} + \frac{t\xi^\mu}{t^\mu\xi}\varepsilon_0 > \frac{\varepsilon_0}{6}.\end{aligned}$$

Consequently,

$$\liminf_{t\to+\infty} \frac{tf^-(x,t)-2F(x,t)}{t^\mu} > 0.$$

Assume $\{u_n\} \subset H_0^1(\Omega)$ such that $\{J(u_n)\}$ is bounded and $(1+\|u_n\|)\lambda(u_n) \to 0$, where $\lambda(u_n) = \min_{w\in\partial J(u_n)} \|w\|$. Then there exists $M > 0$ such that

$$\left|J(u_n)\right| \leqslant M, \qquad \left(1+\|u_n\|\right)\lambda(u_n) \leqslant M.$$

Choose $w_n \in \partial J(u_n)$ such that $\|w_n\| = \lambda(u_n)$. Then there exists $v_n \in \partial g(u_n)$ such that

$$\langle w_n, v\rangle = \int_\Omega \nabla u_n \cdot \nabla v\,dx - \langle v_n, v\rangle, \quad \forall v \in H_0^1(\Omega).$$

Consequently, by (3.5)

$$\begin{aligned}\frac{1}{2}\|u_n\|^2 = J(u_n) + \int_\Omega F\big(x, u_n(x)\big)\,dx &\leqslant M + \int_\Omega \big[c_2\big(u_n^+(x)\big)^r + c_1\big]\,dx \\ &\leqslant c_2 \int_\Omega \big(u_n^+(x)\big)^r\,dx + c_3. \end{aligned} \tag{3.7}$$

Moreover, by (3.6) there exist constants $c_4 > 0$, $\delta_1 > 0$ such that

$$tf^-(x,t) - 2F(x,t) > c_4|t|^\mu, \quad \forall t \geqslant \delta_1.$$

By the locally boundedness of f there exist a constant $c_5 > 0$ such that

$$\big|tf^-(x,t) - 2F(x,t)\big| \leqslant c_5, \quad \forall 0 \leqslant t \leqslant \delta_1.$$

Hence

$$tf^-(x,t) - 2F(x,t) \geqslant c_4|t|^\mu - c_6, \quad \forall t \geqslant 0.$$

Let $g(u) = \int_\Omega F(x, u(x))\,dx$. Consequently,

$$\begin{aligned} 3M \geqslant 2J(u_n) - \langle w_n, u_n\rangle &= -2\int_\Omega F\big(x, u_n(x)\big)\,dx - \langle v_n, -u_n\rangle \\ &\geqslant -2\int_\Omega F\big(x, u_n(x)\big)\,dx - g^0(u_n; -u_n) \\ &\geqslant -2\int_\Omega F\big(x, u_n(x)\big)\,dx + \int_{u_n(x)<0} u_n(x) f^+\big(x, u_n(x)\big)\,dx \\ &\quad + \int_{u_n(x)>0} u_n(x) f^-\big(x, u_n(x)\big)\,dx \\ &= \int_{u_n(x)>0} \big[u_n(x) f^-\big(x, u_n(x)\big) - 2F\big(x, u_n(x)\big)\big]\,dx \\ &\geqslant \int_\Omega \big[c_4\big(u_n^+(x)\big)^\mu - c_6\big]\,dx. \end{aligned}$$

Therefore, $\int_\Omega (u_n^+(x))^\mu\,dx$ is bounded. Since $\mu \geqslant r$, by (3.7), $\|u_n\|$ is bounded. The rest of the proof is the same with proof 2^0 of Theorem 4.3 in [3]. □

Corollary 3.5. *Let $f : \bar\Omega \times R \to R$ be a locally bounded Caratheodory function and conditions* (A_1), (A_3) *and* (A_5) *hold with $q(x) \equiv +\infty$. If there exists $\mu \geqslant r$ such that*

$$\liminf_{t\to+\infty} \frac{tf(x,t) - 2F(x,t)}{|t|^\mu} > 0,$$

then the problem (P_2) *possesses at least one positive solution in $H_0^1(\Omega)$.*

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